PRODUCT SETS AND DELTA-SETS IN AMENABLE GROUPS

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Abstract. Beiglböck, Bergelson and Fish proved that if subsets A,B of a countable discrete amenable group G have positive Banach densities α and β respectively, then the product set AB is piecewise syndetic, i.e. there exists k such that the union of k-many left translates of AB is thick. We give an alternative proof of this result that does not require G to be countable, and moreover yields the explicit bound $k \leq 1/\alpha\beta$. This answers a question that was posed by Beiglböck. We also prove that if $\{A_i\}_{i=1}^n$ are finitely many subsets of G having positive Banach densities α_i and G is countable, then there exists a subset B whose Banach density is at least $\prod_{i=1}^n \alpha_i$ and such that $BB^{-1} \subseteq \bigcap_{i=1}^n A_i A_i^{-1}$. In particular, the latter set is piecewise Bohr.

Introduction.

In 2000 R. Jin proved that the sumset $A + B$ of two sets of integers is piecewise syndetic whenever both A and B have positive Banach density ([10]). Afterwards M. Beiglböck, V. Bergelson and A. Fish generalized Jin's theorem showing that if two subsets A and B of a *countable* amenable group have positive Banach density, then their product set AB is piecewise syndetic, and in fact piecewise Bohr ([2]). In this paper, by using the nonstandard characterization of Banach density in amenable groups, we extend that result to the uncountable case, and we also provide an explicit bound on the number of left translates of AB that are needed to cover a thick set. Moreover, we extend some of the properties of Delta-sets $A-A$ proved in [12] for sets of integers, to the general setting of amenable groups. In particular, applying the pointwise ergodic theorem for countable amenable groups in the nonstandard setting, we show that any finite intersection $\bigcap_{i=1}^n A_i A_i^{-1}$ of sets A_i of positive Banach density contains BB^{-1} for some set B of positive Banach density and, as a consequence, is piecewise Bohr.

Let us now introduce some terminology to be used in the paper, as well as some combinatorial notions that we shall consider. Let G be a group. If $A, B \subseteq G$, we denote by $A^{-1} = \{a^{-1} \mid a \in A\}$ the set of inverses of elements of A. A *translate* of A is a set of the form $xA = \{xa \mid a \in A\}$. More generally, for subsets $A, B \subseteq G$, we denote the *product set* $\{ab \mid a \in A \text{ and } b \in B\}$ by AB.

A set $A \subseteq G$ is k-*syndetic* if k translates of A suffice to cover G, i.e. if $G = FA$ for some $F \subseteq A$ such that $|F| \leq k$. The set A is *thick* if the family of its translates $\{gA \mid g \in G\}$ has the *finite intersection property*, i.e. $\bigcap_{x\in H} xA \neq \emptyset$ for all finite $H \subseteq G$ (equivalently, for every finite H there is $x \in G$ such that $Hx \subseteq A$).

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Another relevant notion is obtained by combining syndeticity and thickness. A set A is *piecewise* k*-syndetic* if k translates of A suffice to cover a thick set, i.e. if FA is thick for some $F \subseteq A$ such that $|F| \leq k$. (For more on these notions see [3].)

Familiarity will be assumed with the basics of *nonstandard analysis*, namely with the notions of *hyperextension* (or nonstandard extension), *internal set*, *hyperfinite set* (or *finite set), the *transfer principle*, and the properties of *overspill* and κ*-saturation* (recall that for every cardinal κ there exist κ-saturated nonstandard models). Moreover, in Section 4 we shall also use the *Loeb measure*. Good references for the nonstandard notions used in this paper are e.g. the introduction given in [4] §4.4, and the monograph [8] where a comprehensive treatment of the theory is given. However, there are several other interesting books on nonstandard analysis and its applications that the reader may also want to consult (see e.g. [1, 6, 5]).

Let us now fix the "nonstandard" notation we shall adopt here. If X is a standard entity, $^*X$ denotes its hyperextension. A subset A of $^*X$ is *internal* if it belongs to the hyperextension of the power set of X. If $\xi, \zeta \in {}^*\mathbb{R}$ are hyperreal numbers, we write $\xi \approx \zeta$ when ξ and ζ are *infinitely close*, i.e. when their distance $|\xi - \zeta|$ is infinitesimal. If $\xi \in {}^*\mathbb{R}$ is finite, then its *shadow* (or *standard part*) $sh(\xi)$ is the unique real number which is infinitely close to ξ. We write $\xi \lesssim \eta$ to mean that $sh(\xi - \eta) \leq 0$, i.e. $\xi < \eta$ or $\xi \approx \eta$.

1. Amenable groups and Banach density

In this paper we aim at generalizing combinatorial properties of sets of integers related to their asymptotic density to more general groups. To this purpose, it is convenient to work in the framework of amenable groups, that are endowed with a suitable notion of density. Amenable groups admit several equivalent characterizations (see e.g. [14, 13]). The most convenient definition for our purposes is the following one, first isolated by Følner [7].

Definition 1.1. *A group G is* amenable *if and only if it satisfies the following*

- Følner's condition*: For every finite $H \subset G$ and for every $\varepsilon > 0$ there exists a finite set K which is "*(H, ε)-invariant*", i.e. K is nonempty and for every $h \in H$ one has*
$$\frac{|hK \triangle K|}{|K|} \ < \ \varepsilon.$$

The Banach density in amenable groups can be defined using the (H, ε)-invariant sets as the "finite approximations" of G. Using almost invariant sets ensures that the notion of density so obtained is invariant by left translations.

Definition 1.2. *Let G be an amenable group. The (upper)* Banach density $d(A)$ *of a subset $A \subseteq G$ is defined as the least upper bound of the set of real numbers α such that for every finite $H \subseteq G$ and for every $\varepsilon > 0$ there exists a finite K which is (H, ε)-invariant and satisfies $\frac{|A \cap K|}{|K|} \geq \alpha$. Similarly, the* lower Banach density $\underline{d}(A)$ *is the least upper bound of the numbers α such that, for some finite subset H of G and some $\varepsilon > 0$, every finite subset K of G which is (H, ε)-invariant satisfies $\frac{|A \cap K|}{|K|} \geq \alpha$.*

It is not difficult to see that if A is piecewise k-syndetic, then $d(A) \geq 1/k$, and if A is k-syndetic then $\underline{d}(A) \geq 1/k$.

We now prove convenient nonstandard characterizations that will be used in the sequel.

Proposition 1.3. *A group G is amenable if and only if in every sufficiently saturated nonstandard model one finds a "*Følner approximation*" of G, i.e. a nonempty hyperfinite set $E \subseteq {}^*G$ such that for all $g \in G$:*

$$\frac{|gE \triangle E|}{|E|} \approx 0.$$

Moreover, if G is amenable, for all $A \subseteq G$ one has:

$$\begin{aligned} d(A) &= \max\left\{ sh\left(\frac{|{}^*A \cap E|}{|E|}\right) \,\middle|\, E \text{ Følner approximation of } G \right\}. \\ \underline{d}(A) &= \min\left\{ sh\left(\frac{|{}^*A \cap E|}{|E|}\right) \,\middle|\, E \text{ Følner approximation of } G \right\}. \end{aligned}$$

Proof. Assume first that G is amenable. For $g \in G$ and $n \in \mathbb{N}$ let

$$\Gamma(g,n) = \left\{ K \subseteq G \text{ finite nonempty} \,\middle|\, \frac{|gK \triangle K|}{|K|} < \frac{1}{n} \right\}.$$

It is readily seen that by Følner's condition the family of all sets $\Gamma(g,n)$ has the finite intersection property. Then, in any nonstandard model that satisfies κ-saturation with $\kappa > \max\{|G|, \aleph_0\}$, the hyperextensions ${}^*\Gamma(g,n)$ have a nonempty intersection, and every $E \in \bigcap\{{}^*\Gamma(g,n) \mid g \in G, n \in \mathbb{N}\}$ is a Følner approximation of G.

Conversely, given $H = \{g_1, \ldots, g_m\} \subseteq G$ and $\varepsilon > 0$, the existence of a nonempty finite (H,ε)-invariant set is proved by applying transfer to the following property, which holds in the nonstandard model: *"There exists a nonempty hyperfinite $E \subseteq {}^*G$ such that $|g_i E \triangle E| < \varepsilon |E|$ for all $i \in \{1, \ldots, m\}$".*

Suppose now that G is amenable and $A \subseteq G$. Consider a Følner approximation E of G and define $\alpha = sh\left(\frac{|{}^*A \cap E|}{|E|}\right)$. If $H = \{g_1, \ldots, g_n\} \subseteq G$ and $\varepsilon > 0$, applying transfer to the statement *"There exist a nonempty finite subset $E \subseteq {}^*G$ such that $|g_i E \triangle E| < \varepsilon |E|$ for every $i = 1, 2, \ldots, n$ and $|{}^*A \cap F| > \alpha |F|$"* one obtains the existence of an (H,ε)-invariant subset K of G such that $|A \cap K| > \alpha |K|$. This shows that

$$d(A) \geq \sup\left\{ \frac{|{}^*A \cap E|}{|E|} \,\middle|\, E \text{ Følner approximation of } G \right\}.$$

It remains to show that the sup is a maximum, and it is equal to $d(A)$. Define for $g \in G$ and $n \in \mathbb{N}$,

$$\Lambda_A(g,n) = \left\{ K \in \Gamma(g,n) \,\middle|\, \frac{|A \cap K|}{|K|} > d(A) - \frac{1}{n} \right\}.$$

It is easily seen that the family of all sets $\Lambda_A(g,n)$ has the finite intersection property. As before, in any nonstandard model that satisfies κ-saturation with $\kappa > \max\{|G|, \aleph_0\}$, the hyperextensions ${}^*\Lambda_A(g,n)$ have a nonempty intersection, and every $E \in \bigcap\{{}^*\Lambda_A(g,n) \mid g \in G, n \in \mathbb{N}\}$ is a Følner approximation of G such that $sh\left(\frac{|{}^*A \cap E|}{|E|}\right) = d(A)$.

The proof of the nonstandard characterization of the lower Banach density is similar and is omitted. □

It is often used in the literature the notion of *Følner sequence*, i.e. a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G such that, for all $g \in G$,

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} \;=\; 0.$$

We remark that, if the above condition holds, then for every finite $H \subset G$ and for every $\varepsilon > 0$ the sets F_n are (H,ε)-invariant for all sufficiently large n. It follows that a *countable* group G is amenable if and only if it admits a Følner sequence. Moreover, in the countable case the Følner density of a set $A \subseteq G$ is characterized as follows:

$$d(A) \;=\; \sup\left\{ \limsup_{n\to\infty} \frac{|A\cap F_n|}{|F_n|} \;\Big|\; (F_n)_{n\in\mathbb{N}} \text{ a Følner sequence} \right\}.$$

It is a well known fact (see for example [2], Remark 1.1), that if $(F_n)_{n\in\mathbb{N}}$ is any Følner sequence and $A \subseteq G$, then there is a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that

$$d(A) = \limsup_{n\in\mathbb{N}} \frac{|A \cap F_n g_n|}{|F_n|}$$

From this, it immediately follows that, when $G = \mathbb{Z}$, the Banach density as defined here coincides with the usual notion of Banach density for sets of integers.

For an extensive treatment of Banach density and its generalizations in the context of semigroups, the reader is referred to [9].

The following notion of density Delta-sets is a generalization of the Delta-sets $A - A = \{a - a' \mid a, a' \in A\}$ of sets of integers.

Definition 1.4. *Let G be an amenable group, and let $\varepsilon \geq 0$. For $A \subseteq G$, the corresponding* ε-density Delta-set *(or* ε-Delta-set *for short) is defined as $\Delta_\varepsilon(A) = \{g \in G \mid d(A \cap gA) > \varepsilon\}$.*

Observe that $\Delta_\varepsilon(A) \subseteq \Delta_0(A) \subseteq AA^{-1}$. We now introduce a notion of embeddability between sets of a group. The idea is to have a suitable partial ordering relation at hand that preserves the finite combinatorial structure of sets.

Definition 1.5. *Let $A, B \subseteq G$. We say that A is* finitely embeddable *in B, and write $A \lhd B$, if every finite subset of A has a right translate contained in B.*

It is immediate from the definitions that A is thick if and only if $G \lhd A$. Finite embeddability admits the following nonstandard characterization.

Proposition 1.6. *Let $A, B \subseteq G$. Then $A \lhd B$ if and only if in every sufficiently saturated nonstandard model one has $A\eta \subseteq {}^*B$ for some $\eta \in {}^*G$.*

Proof. Notice that $A \lhd B$ if and only if the family $\{a^{-1}B \mid a \in A\}$ has the finite intersection property. So, in any nonstandard model that satisfies *κ-saturation* with $\kappa > |A|$, the intersection $\bigcap_{a\in A} a^{-1*}B$ is nonempty. If η is an element of this set, then $A\eta \subseteq {}^*B$. Conversely, suppose that $A\eta \subseteq {}^*B$ for some $\eta \in {}^*G$. If $H = \{h_1, \ldots, h_n\}$ is a finite subset of A, one obtains the existence of an element $x \in G$ such that $Hx \subseteq B$ by *transfer* from the statement: *"There exists $\eta \in {}^*G$ such that $h_i\eta \in {}^*B$ for $i = 1, \ldots, n$ "*. This shows that $A \lhd B$. □

It is easily verified that, if $A \lhd B$, then $AA^{-1} \subseteq BB^{-1}$ and $\Delta_\varepsilon(A) \subseteq \Delta_\varepsilon(B)$ for every $\varepsilon \geq 0$.

2. Combinatorial properties in a nonstandard setting

In this section we prove combinatorial properties in a nonstandard framework that will be used as key ingredients in the proofs of our main results. The first one below can be seen as a form of *pigeonhole principle* that holds in a hyperfinite setting.

Lemma 2.1. *Let E be a hyperfinite set, and let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a finite family of internal subsets of E. Assume that γ, ε are non-negative real numbers such that*

- $\varepsilon < \gamma^2$ *;*
- $sh\left(\frac{|C_\lambda|}{|E|}\right) \geq \gamma$ *for every $\lambda \in \Lambda$;*
- $|\Lambda| > \frac{\gamma-\varepsilon}{\gamma^2-\varepsilon}$.

Then there exist distinct $\lambda, \mu \in \Lambda$ such that

$$sh\left(\frac{|C_\lambda \cap C_\mu|}{|E|}\right) > \varepsilon.$$

Proof. Without loss of generality, let us assume that $\mathrm{sh}(|C_\lambda|/|E|) = \gamma$ for every $\lambda \in \Lambda$. Suppose by contradiction that for all distinct $\lambda \neq \mu$:

$$\mathrm{sh}\left(\frac{|C_\lambda \cap C_\mu|}{|E|}\right) \leq \varepsilon.$$

For $i \in E$, set $a_i = \sum_{i\in\Lambda} \chi_\lambda(i)$ where $\chi_\lambda : E \to \{0,1\}$ denotes the characteristic function of C_λ. Observe that

$$\sum_{i\in E} a_i = \sum_{\lambda\in\Lambda} |C_\lambda|$$

and

$$\sum_{i\in E} a_i^2 = \sum_{\lambda\in\Lambda} |C_\lambda| + \sum_{\lambda\neq\mu} |C_\lambda \cap C_\mu| \, .$$

If we set $b_i = 1$, then by the *Cauchy-Schwartz inequality*,

$$\begin{aligned}\left(\sum_{\lambda\in\Lambda}|C_\lambda|\right)^2 &= \left(\sum_{i\in E}a_ib_i\right)^2\\ &\le \left(\sum_{i\in E}a_i^2\right)\left(\sum_{i\in E}b_i^2\right)\\ &= |E|\left(\sum_{\lambda\in\Lambda}|C_\lambda| + \sum_{\lambda\neq\mu}|C_\lambda\cap C_\mu|\right).\end{aligned}$$

Dividing by $|E|^2$, one gets

$$\begin{aligned}|\Lambda|^2\gamma^2 &\approx \left(\sum_{\lambda\in\Lambda}\frac{|C_\lambda|}{|E|}\right)^2\\ &\le \sum_{\lambda\in\Lambda}\frac{|C_\lambda|}{|E|} + \sum_{\lambda\neq\mu}\frac{|C_\lambda\cap C_\mu|}{|E|}\\ &\approx |\Lambda|\gamma + \sum_{\lambda\neq\mu}\frac{|C_\lambda\cap C_\mu|}{|E|}.\end{aligned}$$

As there are $|\Lambda|(|\Lambda|-1)$ ordered pairs (λ,μ) such that $\lambda\neq\mu$, we get

$$\varepsilon|\Lambda|(|\Lambda|-1) \gtrsim \sum_{\lambda\neq\mu}\frac{|C_\lambda\cap C_\mu|}{|E|} \gtrsim |\Lambda|\gamma(|\Lambda|\gamma-1).$$

Dividing by $|\Lambda|$, we obtain that $|\Lambda|\gamma^2 \le \gamma + \varepsilon(|\Lambda|-1)$, and finally:

$$|\Lambda| \le \frac{\gamma-\varepsilon}{\gamma^2-\varepsilon}.$$

This contradicts our assumptions and concludes the proof. □

Recall that we called *Følner approximation* of G any nonempty hyperfinite set $E\subseteq {}^*G$ such that for all $g\in G$:

$$\frac{|gE\triangle E|}{|E|}\approx 0.$$

Lemma 2.2. *Let E be a Følner approximation of G, and suppose that $C\subseteq {}^*G$ is such that $st\left(\frac{|C\cap E|}{|E|}\right)=\gamma>0$. Let $0\le\varepsilon<\gamma^2$ and $k=\left\lfloor\frac{\gamma-\varepsilon}{\gamma^2-\varepsilon}\right\rfloor$. Define*

$$\mathcal{D}_\varepsilon^E(C)=\left\{g\in G\,\middle|\, st\left(\frac{|C\cap gC\cap E|}{|E|}\right)>\varepsilon\right\}.$$

Then, for every $P\subseteq G$ and every $g_0\in P$ there exists $F\subseteq P$ such that $g_0\in F$, $|F|\le k$ and $P\subseteq F\cdot\mathcal{D}_\varepsilon^E(C)$.

Proof. We define elements g_i of P by recursion. Suppose that g_i has been defined for $0\le i<n$. If $P\subseteq\{g_0,\ldots,g_{n-1}\}\cdot\mathcal{D}_\varepsilon^E(C)$, then set $g_n=g_{n-1}$. Otherwise, pick

$$g_n\in P\setminus\left(\{g_0,\ldots,g_{n-1}\}\cdot\mathcal{D}_\varepsilon^E(C)\right).$$

We claim that, $g_k = g_{k-1}$, i.e. $P \subseteq \{g_0, \ldots, g_{k-1}\} \cdot \mathcal{D}^E_\varepsilon(C)$. Suppose by contradiction that this is not the case. Then, for every $i < j < k$, we have

$$g_j \notin \{g_0, \ldots, g_i\} \cdot \mathcal{D}^E_\varepsilon(C).$$

This implies that $g_i^{-1} g_j \notin \mathcal{D}^E_\varepsilon(C)$ and

$$\varepsilon \ \geq \ \mathrm{st}\left(\frac{\left|C \cap g_i^{-1} g_j C \cap E\right|}{|E|}\right) \ = \ \mathrm{st}\left(\frac{|g_i C \cap g_j C \cap E|}{|E|}\right).$$

By the previous lemma applied to the family $\{g_i C \cap E \mid i < k\}$, there exist $i < j < k$ such that

$$\frac{|g_i C \cap g_j C \cap E|}{|E|} > \varepsilon.$$

This is a contradiction. $\square$

Lemma 2.3. *Let $U, V \subseteq {}^*G$ be hyperfinite sets, and let $C \subseteq U$ and $D \subseteq V$ be internal subsets. Then there exists $\zeta, \vartheta \in U$ such that*

$$(1) \qquad \frac{|D\zeta \cap C|}{|V|} \ \geq \ \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \max_{d \in D} \frac{|dU \bigtriangleup U|}{|U|}.$$

$$(2) \qquad \frac{|\vartheta D \cap C|}{|V|} \ \geq \ \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \max_{d \in D} \frac{|Ud \bigtriangleup U|}{|U|}.$$

Proof. Let $\chi_C : U \to \{0, 1\}$ be the characteristic function of C. For $d \in D$, one has

$$\frac{1}{|U|} \sum_{u \in U} \chi_C(du) \ = \ \frac{|C \cap dU|}{|U|} \ = \ \frac{|C| - |C \cap (U \setminus dU)|}{|U|} \ \geq \ \frac{|C|}{|U|} - \frac{|dU \bigtriangleup U|}{|U|}.$$

Then,

$$\begin{aligned}
\frac{1}{|U|} \sum_{u \in U} \frac{|Du \cap C|}{|V|} \quad &= \quad \frac{1}{|U|} \sum_{u \in U} \left(\frac{1}{|V|} \sum_{d \in D} \chi_C(du) \right) \\
&= \quad \frac{1}{|V|} \sum_{d \in D} \left(\frac{1}{|U|} \sum_{u \in U} \chi_C(du) \right) \\
&\geq \quad \frac{1}{|V|} \sum_{d \in D} \left(\frac{|C|}{|U|} - \frac{|dU \bigtriangleup U|}{|U|} \right) \\
&\geq \quad \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \max_{d \in D} |dU \bigtriangleup U| / |U|.
\end{aligned}$$

Thus for some $\zeta \in U$,

$$\frac{|D\zeta \cap C|}{|V|} \ \geq \ \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \max_{d \in D} \frac{|dU \bigtriangleup U|}{|U|}.$$

The second part of the statement is obtained applying the first part to the opposite group of G (which is amenable as well). $\square$

Lemma 2.4. *Let G be an amenable group. Suppose $A_0, A_1, \ldots, A_n \subseteq G$ are subsets with Banach densities $d(A_i) \geq \alpha_i$. Then in every sufficiently saturated nonstandard model there exist Følner approximations $E, F \subseteq {}^*G$ and elements $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in {}^*G$ such that*

$$\frac{|{}^*A_0 \,\cap\, (\bigcap_{i=1}^{n} {}^*A_i\, \xi_i) \,\cap\, E|}{|E|} \;\gtrsim\; \prod_{i=0}^{n} \alpha_i. \tag{1}$$

$$\frac{|{}^*A_0 \,\cap\, \left(\bigcap_{i=1}^{n} \eta_i\, {}^*A_i^{-1}\right) \,\cap\, F|}{|F|} \;\gtrsim\; \prod_{i=0}^{n} \alpha_i. \tag{2}$$

Proof. We proceed by induction. Let us start with property (1). The base $n = 0$ is given by the nonstandard characterization of Banach density. Now let the subsets $A_0, A_1, \ldots, A_{n+1} \subseteq G$ be given where $d(A_i) \geq \alpha_i$. By the inductive hypothesis there exist a Følner approximation $V \subseteq {}^*G$ and elements $\xi_1, \ldots, \xi_n \in {}^*G$ that satisfy $|{}^*A_0 \cap (\bigcap_{i=1}^{n} {}^*A_i\, \xi_i) \cap V|/|V| \gtrsim \prod_{i=0}^{n} \alpha_i$. We now want to find a Følner approximation U that witnesses $d(A_{n+1}) \geq \alpha_{n+1}$ and with the additional feature of being "almost invariant" with respect to left translates by elements in V. To this purpose, pick a hyperfinite $V' \supseteq V \cup G$ (notice that this is possible by κ-saturation with $\kappa > |G|$). Consider the following property that directly follows from the definition of Følner density: "*For every $k \in \mathbb{N}$ and every finite $H \subseteq G$ there exists a nonempty finite $K \subseteq G$ which is $(H, 1/k)$-invariant and such that the relative density $|A_{n+1} \cap K|/|K| > \alpha_{n+1} - 1/k$*" . If $\nu \in {}^*\mathbb{N}$, by *transfer* we get a nonempty hyperfinite $U \subseteq {}^*G$ that is $\left(V', \frac{1}{\nu}\right)$-invariant (and, in particular, is a Følner approximation of G) and such that

$$\frac{|{}^*A_{n+1} \cap U|}{|U|} > \alpha_{n+1} - \frac{1}{\nu} \approx \alpha_{n+1}.$$

Now apply (1) of the previous lemma to the internal sets $C = {}^*A_{n+1} \cap U \subseteq U$ and $D = {}^*A_0 \cap (\bigcap_{i=1}^{n} {}^*A_i\, \xi_i) \cap V \subseteq V$, and pick an element $\zeta \in U$ such that

$$\frac{|D\zeta \cap C|}{|V|} \;\geq\; \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \max_{d \in D} \frac{|dU \bigtriangleup U|}{|U|} \;\geq\; \frac{|C|}{|U|} \cdot \frac{|D|}{|V|} - \frac{1}{\nu} \;\gtrsim\; \prod_{i=0}^{n+1} \alpha_i.$$

This yields the conclusion with $E = V$. In fact, by letting $\xi_{n+1} = \zeta^{-1}$

$$\frac{|D\zeta \cap C|}{|V|} \leq \frac{|D\zeta \cap {}^*A_{n+1}|}{|V|} = \frac{|D \cap {}^*A_{n+1}\xi_{n+1}|}{|V|} = \frac{\left|{}^*A_0 \,\cap\, \left(\bigcap_{i=1}^{n+1} {}^*A_i\, \xi_i\right) \,\cap\, V\right|}{|V|}.$$

As for (2), we proceed in a similar way as above by considering sets of inverses. Precisely, let $V \subseteq {}^*G$ be a Følner approximation of G and $\eta_1, \ldots, \eta_n$ be elements of ${}^*G$ that satisfy

$$\frac{|{}^*A_0 \,\cap\, \left(\bigcap_{i=1}^{n} \eta_i\, {}^*A_i^{-1}\right) \,\cap\, V|}{|V|} \;\gtrsim\; \prod_{i=0}^{n} \alpha_i.$$

Pick a Følner approximation U that witnesses $d(A_{n+1}) \geq \alpha_{n+1}$ and with the additional feature of being "almost invariant" with respect to translates by elements in the set of inverses V^{-1}, i.e. $|{}^*A_{n+1} \cap U|/|U| \gtrsim \alpha_{n+1}$ and $|xU \bigtriangleup U|/|U| \approx 0$ for all $x \in V^{-1}$. Then apply (2) of the previous lemma to the internal sets

$C = {}^*A_{n+1}^{-1} \cap U^{-1} \subseteq U^{-1}$ and $D = {}^*A_0 \cap (\bigcap_{i=1}^n \eta_i\, {}^*A_i^{-1}) \cap V \subseteq V$, and get the existence of an element $\vartheta \in U^{-1}$ such that

$$\begin{aligned}\frac{|\vartheta D \cap C|}{|V|} &\geq \frac{|C|}{|U^{-1}|}\cdot\frac{|D|}{|V|} - \max_{d\in D}\frac{|U^{-1}d \triangle U^{-1}|}{|U^{-1}|}\\ &= \frac{|C^{-1}|}{|U|}\cdot\frac{|D|}{|V|} - \max_{d\in D^{-1}}\frac{|dU \triangle U|}{|U|}\\ &\gtrsim \prod_{k=1}^{n+1}\alpha_k.\end{aligned}$$

Since

$$\frac{|\vartheta D \cap C|}{|V|} \leq \frac{|\vartheta D \cap {}^*A_{n+1}^{-1}|}{|V|} = \frac{|D \cap \vartheta^{-1}{}^*A_{n+1}^{-1}|}{|V|} = \frac{\left|{}^*A_0 \cap \left(\bigcap_{i=0}^{n+1}\eta_i{}^*A_{n+1}^{-1}\right) \cap V\right|}{|V|},$$

the statement is proved by letting $F = V$ and $\eta_{n+1} = \vartheta^{-1}$. □

3. Intersection properties of Delta-sets and Jin's theorem

The nonstandard lemmas of the previous section entail a general result about intersections of density Delta-sets.

Theorem 3.1. *Suppose that, for $i \leq n$, A_i is a subset of G of positive Banach density α_i. Let $0 \leq \varepsilon < \beta^2$ where $\beta = \prod_{i=1}^n \alpha_i$, $P \subseteq G$ and $g_0 \in P$. If $r = \left\lfloor \frac{\beta-\varepsilon}{\beta^2-\varepsilon}\right\rfloor$, then there exists a finite $L \subseteq G$ such that $|L| \leq r$, $g_0 \in L$ and $P \subseteq L\cdot(\bigcap_{i=1}^n \Delta_\varepsilon(A_i))$.*

Proof. By Lemma 2.4 where $A_0 = G$, we can pick a Følner approximation $E \subseteq {}^*G$ and elements $\xi_1, \ldots, \xi_n \in {}^*G$ such that

$$\frac{|(\bigcap_{i=1}^n {}^*A_i\xi_i) \cap E|}{|E|} \gtrsim \beta.$$

Define the internal set $C = \bigcap_{i=1}^n {}^*A_i\xi_i$ and observe that

$$\mathcal{D}_\varepsilon^E(C) \subseteq \bigcap_{i=1}^n \Delta_\varepsilon({}^*A_i).$$

To see this, notice that if $g \in \mathcal{D}_\varepsilon^E(C)$ then for every $j = 1, \ldots, n$:

$$\begin{aligned}\varepsilon &< \frac{|C \cap gC \cap E|}{|E|} = \frac{|(\bigcap_{i=1}^n {}^*A_i\xi_i) \cap g\,(\bigcap_{i=1}^n {}^*A_i\xi_i) \cap E|}{|E|}\\ &\leq \frac{|{}^*A_j\xi_j \cap g\,{}^*A_j\xi_j \cap E|}{|E|} = \frac{|{}^*(A_j \cap gA_j) \cap E\xi_j^{-1}|}{|E\xi_j^{-1}|}.\end{aligned}$$

Now apply Lemma 2.2 to C and get a finite $L \subseteq P$ such that $|L| \leq r$, $g_0 \in L$ and

$$P \subseteq L \cdot \mathcal{D}_\varepsilon^E(C) \subset \bigcap_{i=1}^n \Delta_\varepsilon\left({}^*A_i\right).$$

□

By applying Theorem 3.1 where $P = G$, one immediately obtains the following

Corollary 3.2. *Under the assumptions of Theorem 3.1, $\bigcap_{i=1}^{n} \Delta_\varepsilon(A_i)$ is r-syndetic and, as a consequence, its lower Banach density is at least $1/r$.*

For $k \in \mathbb{N}$, denote by

- $A^{(k)} = \{g^k \mid g \in A\}$ the set of k-th powers of elements of A.
- $\sqrt[k]{A} = \{g \in G \mid g^k \in A\}$ the set of k-th roots of elements of A.

Corollary 3.3. *Under the assumptions of Theorem 3.1, for every $k \in \mathbb{N}$ the intersection $\bigcap_{i=1}^{n} \sqrt[k]{\Delta_\varepsilon(A_i)}$ is r-syndetic and, as a consequence, its lower Banach density is at least $1/r$.*

Proof. Apply Theorem 3.1 with $P = G^{(k)}$ and get the existence of a finite set $L \subseteq G^{(k)}$ such that $|L| \leq \frac{\beta-\varepsilon}{\beta^2-\varepsilon}$ and $G^{(k)} \subseteq L \cdot (\bigcap_{i=1}^{n} \Delta_\varepsilon(A_i))$. Pick $H \subseteq G$ such that $|H| = |L|$ and $H^{(k)} = L$. Then for every $g \in G$ one has $g^k = h^k \cdot x$ for suitable $h \in H$ and $x \in \bigcap_{i=1}^{n} \Delta_\varepsilon(A_i)$. Equivalently, for every $g \in G$ there exists $h \in H$ such that $(h^{-1}g)^k \in \bigcap_{i=1}^{n} \Delta_\varepsilon(A_i)$, and hence

$$h^{-1}g \in \bigcap_{i=1}^{n} \sqrt[k]{\Delta_\varepsilon\left(A_i\right)}.$$

This shows that $G = H \cdot \left(\bigcap_{i=1}^{n} \sqrt[k]{\Delta_\varepsilon(A_i)}\right)$. □

Next, we prove the existence of an explicit bound in Jin's theorem that only depends on the densities of the given sets.

Theorem 3.4. *Let G be an amenable group. If $X \subseteq G$ is infinite, $w \in X$ and $A, B \subseteq G$ have positive Banach densities $d(A) = \alpha$ and $d(B) = \beta$ respectively, then there exists a finite $F \subset X$ such that:*

- *$w \in F$;*
- *$|F| \leq \frac{1}{\alpha\beta}$;*
- *$X \lhd FAB$.*

Proof. By Lemma 2.4, we can pick a Følner approximation $E \subseteq {}^*G$ and an element $\eta \in {}^*G$ such that the internal set $X = {}^*A \cap \eta^*B^{-1} \cap E$ has relative density $|X|/|E| \gtrsim \alpha\beta$. Then by Lemma 2.2 with $\varepsilon = 0$, there exists a finite $F \subset G$ such that $|F| \leq 1/\alpha\beta$ and $X \subseteq F \cdot \mathcal{D}_0^E(X)$. If $g \in X$, there are $\xi \in F$ and $y \in \mathcal{D}_0^E(X)$ such that $g = \xi y$. Since $y \in \mathcal{D}_0^E(X)$, ${}^*A \cap \eta^*B^{-1} \cap y^*A \cap y\eta^*B^{-1} \neq \emptyset$. In particular, $y = ab\eta^{-1}$ for some $a \in {}^*A$ and $b \in {}^*B$. Therefore,

$$g = \xi y = \xi ab\eta^{-1}$$

and

$$g\eta = \xi ab \in F^*A^*B = {}^*(FAB)\,.$$

Since this is true for every $g \in X$,

$$X\eta \subset {}^*(FAB)\,.$$

Hence, by the nonstandard characterization of finite embeddability,

$$X \lhd FAB.$$ □

Corollary 3.5. *Under the hypothesis of Theorem 3.4, AB is piecewise k-syndetic where $k = \left\lfloor \frac{1}{\alpha\beta} \right\rfloor$.*

Proof. Set $X = G$ and apply Theorem 3.4. Thus, $G \lhd FAB$ for some $F \subset G$ such that $|F| \leq \frac{1}{\alpha\beta}$. Henceforth FAB is thick and AB is piecewise $\left\lfloor \frac{1}{\alpha\beta} \right\rfloor$-syndetic. □

4. Countable amenable groups

Throughout this section we focus on *countable* amenable groups and prove finite embeddability properties.

By [2] (Corollary 5.3), if $A \subseteq G$ has positive Følner density and G is countable, then AA^{-1} is piecewise Bohr. Moreover, by [2] (Lemma 5.4), if $A, B \subseteq G$, A is piecewise Bohr and $A \lhd B$, then B is piecewise Bohr as well. It is a standard result in ergodic theory (see for example [11]) that any countable discrete amenable group G has a Følner sequence $(F_n)_{n\in\mathbb{N}}$ for which the *pointwise ergodic theorem* holds. This means that, if G acts on a probability space $(X, \mathcal{B}, \mu)$ by measure preserving transformations and $f \in L^1(\mu)$, then there is a G-invariant $\bar{f} \in L^1(\mu)$ such that, for μ-almost all $x \in X$:

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) \;=\; \bar{f}(x)\,.$$

Lemma 4.1. *If E is a Følner approximation of G, $0 < \gamma \leq 1$, and $C \subseteq E$ is such that $\frac{|C|}{|E|} \gtrsim \gamma$, then there exists $\xi \in E$ such that*

$$d\left(C\xi^{-1} \cap G\right) \geq \gamma.$$

Proof. Pick a Følner sequence $(F_n)_{n\in\mathbb{N}}$ for G that satisfies the pointwise ergodic theorem. Consider the (separable) σ-algebra $\mathcal{B}$ on E generated by the characteristic function χ_C of C, the probability space $(E, \mathcal{B}, \mu)$ where μ is the restriction of the Loeb measure to $\mathcal{B}$ and the measure preserving action of G on $(E, \mathcal{B}, \mu)$ by left translations. Since χ_C belongs to $L^1(\mu)$, there is a G-invariant function $\bar{f} \in L^1(\mu)$ such that the sequence

$$\left(\frac{1}{|F_n|} \sum_{g\in F_n} \chi_C(gx)\right)_{n\in\mathbb{N}}$$

converges to $\bar{f}(x)$ for μ-a.a. $x \in E$ and hence, by the Lebesgue dominated convergence theorem, in $L^1(\mu)$. This implies in particular that

$$\int \bar{f} d\mu = \int \chi_C d\mu = st\left(\frac{|C|}{|E|}\right) = \gamma.$$

Thus, the set of $x \in E$ such that $\bar{f}(x) \geq \gamma$ is non negligible and, in particular, there is $\xi \in X$ such that $\bar{f}(\xi) \geq \gamma$ and the sequence

$$\left(\frac{1}{|F_n|} \sum_{g\in F_n} \chi_C(g\xi)\right)_{n\in\mathbb{N}}$$

converges to $\bar{f}(\xi) \geq \gamma$. Observe now that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{|F_n|} \sum_{g \in F_n} \chi_C(g\xi) &= \frac{|C \cap F_n \xi|}{|F_n|} \\ &= \frac{\left|C\xi^{-1} \cap F_n\right|}{|F_n|} \\ &= \frac{\left|\left(C\xi^{-1} \cap G\right) \cap F_n\right|}{|F_n|}. \end{aligned}$$

From this and the fact $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence for G it follows that

$$d\left(C\xi^{-1} \cap G\right) \geq \gamma. \qquad \square$$

Theorem 4.2. *Let G be a countable amenable group and suppose that $A_1, \ldots, A_n \subseteq G$ have positive Banach densities $d(A_i) = \alpha_i$. Then there exists $B \subseteq G$ such that $d(B) \geq \prod_{i=1}^n \alpha_i$ and $B \lhd A_i$ for every $i = 1, \ldots, n$. As a consequence, $BB^{-1} \subseteq \bigcap_{i=1}^n A_i A_i^{-1}$ and $\Delta_\varepsilon(B) \subseteq \bigcap_{i=1}^n \Delta_\varepsilon(A_i)$ for every $\varepsilon \geq 0$. In particular, $\bigcap_{i=1}^n A_i A_i^{-1}$ is piecewise Bohr.*

Proof. By Lemma 2.4 there exist a Følner approximation $E \subseteq {}^*G$ and elements $\xi_1, \ldots, \xi_n \in {}^*G$ such that

$$\frac{\left|{}^*A_0 \cap \bigcap_{i=1}^n {}^*A_i \xi_i \cap E\right|}{|E|} \gtrsim \prod_{i=0}^n \alpha_i.$$

By applying 4.1 to E and $C = {}^*A_0 \cap \bigcap_{i=1}^n \xi_i {}^*A_i \cap E$ one obtains $\eta \in E$ such that

$$d\left(C\eta^{-1} \cap G\right) \geq \prod_{i=0}^n \alpha_i.$$

Define $B = C\eta^{-1} \cap G$ and observe that $B\eta \subseteq {}^*A_0$ and $B\eta\xi_i^{-1} \subseteq {}^*A_i$ for $1 \leq i \leq n$. This implies that $B \lhd A_i$ for $0 \leq i \leq n$. $\square$

5. Final remarks and open problems

In a preliminary version of [10], R. Jin asked whether one could estimate the number k needed to have $A + B + [0, k)$ thick (under the assumption that both sets $A, B \subseteq \mathbb{N}$ have positive Banach density). In the final published version of that paper, he then pointed out that no such estimate for k exists which depends only on the densities of A and B. In fact, the following holds.

- *Let $\alpha, \beta > 0$ be real numbers such that $\alpha + \beta < 1$, and let $k \in \mathbb{N}$. Then there exist sets $A_k, B_k \subseteq \mathbb{N}$ such that the asymptotic densities $d(A_k) > \alpha$ and $d(B_k) > \beta$ but $A_k + B_k + [0, k)$ is not thick.*

An example can be constructed as follows.[1] Pick natural numbers M, N, L such that $M/L > \alpha$, $N/L > \beta$, and $M/L + N/L + 1/L < 1$. For every $k \in \mathbb{N}$, consider

[1] This example did not appear in [10], and it is reproduced here with Jin's permission.

the following subsets of $\mathbb{N}$:

$$A_k = \bigcup_{n=0}^{\infty}[Lnk, Lnk+Mk) \quad \text{and} \quad B_k = \bigcup_{n=0}^{\infty}[Lnk, Lnk+Nk).$$

Then the following properties are verified in a straightforward manner:

- The asymptotic densities $d(A_k) = M/L$ and $d(B_k) = N/L$.
- $A_k + B_k + [0,k) = \bigcup_{n=0}^{\infty}[Lnk, Lnk + Mk + Nk + k)$.

Since $Lkn+Mk+Nk+k < Lkn+Lk = Lk(n+1)$, it follows that $A_k+B_k+[0,k)$ is not thick, as it consists of disjoint intervals of length $(M+N+1)k$.

However, as remarked by M. Beiglböck, the problem was left open if one replaces the length k of the interval $[0,k)$ with the cardinality k of an arbitrary finite set. As shown by our Theorem 3.4, one can in fact give the bound $k \le 1/\alpha\beta$. Now, the question naturally arises as to whether such a bound is optimal.

Next, it is easy to see that if G is abelian and $B \subseteq G$ then $d(B) = d\left(B^{-1}\right)$. Thus, it follows from Corollary 3.5 that if $A, B \subseteq G$ are such that $d(A) = \alpha$ and $d(B) = \beta$ and G is abelian then both AB and AB^{-1} are piecewise $\left\lfloor \frac{1}{\alpha\beta} \right\rfloor$-syndetic. It would be interesting to know if the same is true for more general amenable groups. More precisely: if G is an amenable group and $B \subseteq G$, then do B and B^{-1} have the same density? Or at least, is it always the case that B has positive density if and only if B^{-1} has positive density? Besides, is the statement of Corollary 3.5 still true where one replaces AB with AB^{-1}?

Finally, all the results of this paper are proved without assumptions on the cardinality of the group, apart from Theorem 4.2, where G is supposed to be countable. It would be interesting to know if also this result holds for any amenable group, regardless of its cardinality.

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